# Math 307 - Differential Equations
## Homework Solutions
## Spring 2017

# 1. Homework 1 (2/7 & 2/9)

## Section 0.3 (My Book).

**Exercise 8.** *Compute*

$$\int e^x \sin x \; dx$$

*Solution.* Use integration by parts right away with $u = e^x$ and $dv = \sin x \; dx$ to get:

$$\int e^x \sin x \; dx = -e^x \cos x + \int e^x \cos x \; dx.$$

Repeat this again on the second integral with $u = e^x$ and $dv = \cos x \; dx$ to get:

$$\int e^x \sin x \; dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \; dx.$$

Move the integral to the left side:

$$2 \int e^x \sin x \; dx = -e^x \cos x + e^x \sin x.$$

Divide by 2 and add the constant of integration to the right side:

$$\int e^x \sin x \; dx = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C.$$

□

## Section 0.6 (My Book).

**Exercise 4.** *Verify Clairaut's theorem for*

$$f(x, y) = xe^{2y}$$

*Solution.*

$$\begin{aligned} \frac{\partial f}{\partial x} &= e^{2y} \\ \frac{\partial^2 f}{\partial y \partial x} &= 2e^{2y} \\ \frac{\partial f}{\partial y} &= 2xe^{2y} \\ \frac{\partial^2 f}{\partial x \partial y} &= 2e^{2y} \end{aligned}$$

so we've verified that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

□

**Exercise 7.** *Integrate the following function with respect of all of its variables*

$$g(x, y, z) = xz^2 e^{xy}$$

*Solution.* The integral with respect to $x$ requires integration by parts. Use $u = xz^2$ and $dv = e^{xy}dx$ to get

$$\int xz^2 e^{xy} dx = \frac{xz^2}{y} e^{xy} - \int \frac{z^2}{y} e^{xy} dx = \frac{xz^2}{y} e^{xy} - \frac{z^2}{y^2} e^{xy} + g(y, z).$$

The other two integrals are more straightforward:

$$\begin{aligned} \int xz^2 e^{xy} dy &= z^2 e^{xy} + g(x, z) \\ \int xz^2 e^{xy} dz &= \frac{1}{3} xz^3 e^{xy} + g(x, y) \end{aligned}$$

□

# Section 1.2.

**Exercise 4.** *Solve the given initial value problem.*

(b) $y' = x \sin x^2,\ y\left(\sqrt{\frac{\pi}{2}}\right) = 1$
(d) $y'' = x^4,\ y(2) = -1,\ y'(2) = -1$
(i) $y''' = 2x + 1,\ y(2) = 1,\ y'(2) = -4,\ y''(2) = 7$

*Solution.*

(b) Begin by integrating ($u$-sub with $u = x^2$)

$$y = -\frac{1}{2} \cos x^2 + C,$$

and use the initial value

$$y\left(\sqrt{\frac{\pi}{2}}\right) = -\frac{1}{2} \cos \frac{\pi}{2} + C = C = 1$$

So $C = 1$ and the solution is

$$y = -\frac{1}{2} \cos x^2 + 1.$$

(d) Integrate

$$y' = \frac{1}{5} x^5 + C_1,$$

and use the initial value for $y'$

$$y'(2) = \frac{1}{5}(2)^5 + C_1 = \frac{32}{5} + C_1 = -1 \quad \Longrightarrow \quad C_1 = -\frac{37}{5}.$$

Then

$$y' = \frac{1}{5} x^5 - \frac{37}{5}.$$

Now integrate again

$$y = \frac{1}{30} x^6 - \frac{37}{5} x + C_2$$

and use the initial value for $y$

$$y(2) = \frac{64}{30} - \frac{74}{5} + C_2 = -\frac{190}{15} + C_2 = -1 \quad \Longrightarrow \quad C_2 = \frac{175}{15} = \frac{35}{3}$$

Thus the final answer is

$$y = \frac{1}{30}x^6 - \frac{37}{5}x + \frac{35}{3}.$$

(i) Integrate

$$y'' = x^2 + x + C,$$

use the initial value for $y''$

$$y''(2) = 4 + 2 + C = 6 + C = 7 \quad \Longrightarrow \quad C_1 = 1$$

then

$$y'' = x^2 + x + 1.$$

Integrate again

$$y' = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C_2$$

use the initial value for $y'$

$$y'(2) = \frac{8}{3} + 2 + 2 + C_2 = \frac{20}{3} + C_2 = -4 \quad \Longrightarrow \quad C_2 = -4 - \frac{20}{3} = -\frac{32}{3}$$

then

$$y' = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x - \frac{32}{3}.$$

Finally, integrate one more time

$$y = \frac{1}{12}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{32}{3}x + C_3$$

plug in the initial value for $y$

$$y = \frac{4}{3} + \frac{4}{3} + 2 - \frac{64}{3} + C_3 = -\frac{50}{3} + C_3 = 1 \quad \Longrightarrow \quad C_3 = \frac{53}{3}.$$

So, the final answer is

$$y = \frac{1}{12}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{32}{3}x + \frac{53}{3}.$$

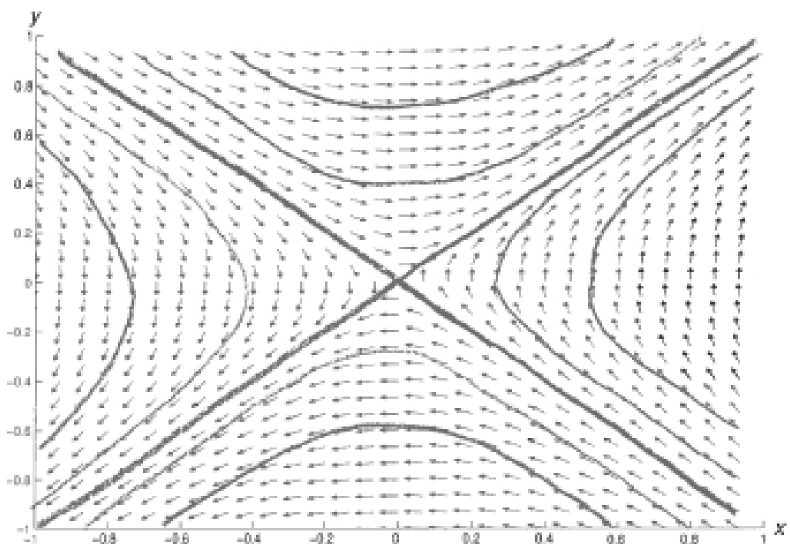
□

# Section 1.3.

**Exercise 1.** *Sketch some integral curves in the given direction field for*

$$y' = \frac{x}{y}.$$

*Solution.* Here's some sketched integral curves.



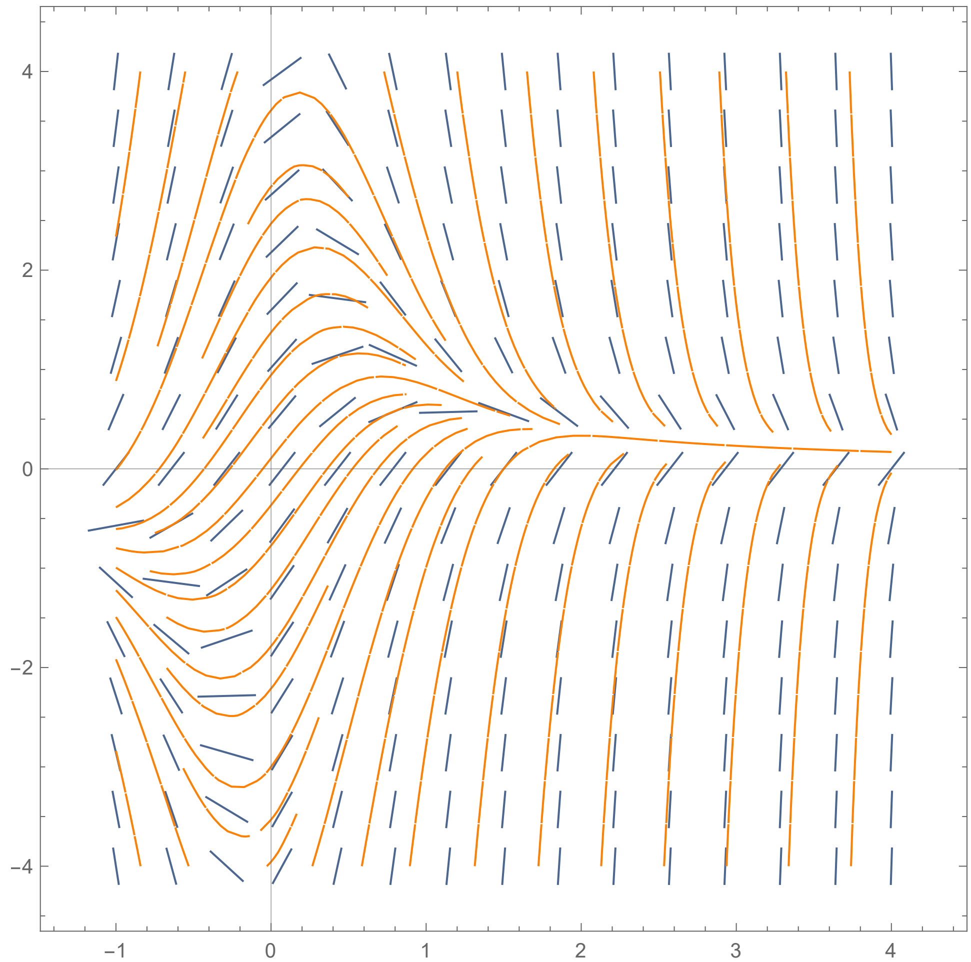
□

**Exercise 13.** *Construct a direction field and plot some integral curves in the indicated rectangular region*

$$y' = 2 - 3xy; \{-1 \leq x \leq 4, -4 \leq y \leq 4\}.$$

*Solution.* Here's *Mathematica* code that will graph the image below

```
Show[VectorPlot[{1, 2 - 3 x y}, {x, -1, 4}, {y, -4, 4},
  VectorStyle -> Arrowheads[0], Axes -> True,
  VectorScale -> {Tiny, Automatic, None}
  ], StreamPlot[{1, 2 - 3 x y}, {x, -1, 4}, {y, -4, 4},
  StreamStyle -> {Orange, Arrowheads[0]}
  ]]
```


4
2
0
−2
−4
−1
0
1
2
3
4

□

# 2. Homework 2 (2/14 & 2/16)

## Section 2.1.

**Exercise 16.** *Find the general solution of*

$$y' + \frac{1}{x}y = \frac{7}{x^2} + 3.$$

*Solution.* This is a linear equation with $p(x) = \frac{1}{x}$ and $q(x) = \frac{7}{x^2} + 3$. The integrating factor is

$$\mu(x) = e^{\int^x \frac{1}{s} ds} = e^{\ln x} = x$$

and so

$$\begin{aligned} y &= \frac{1}{x}\int x\left(\frac{7}{x^2} + 3\right) dx \\ &= \frac{1}{x}\int \left(\frac{7}{x} + 3x\right) dx \\ &= \frac{1}{x}\left(7\ln|x| + \frac{3}{2}x^2 + C\right) \\ &= \frac{7\ln|x|}{x} + \frac{3}{2}x + \frac{C}{x} \end{aligned}$$

□

## Section 2.2.

**Exercise 17.** *Solve the initial value problem and find the interval of validity of the solution for*

$$y'(x^2 + 2) + 4x(y^2 + 2y + 1) = 0,\ \ y(1) = -1.$$

*Solution.* Separate the equation:

$$y'(x^2 + 2) + 4x(y^2 + 2y + 1) = 0 \implies y'(x^2 + 2) = -4x(y^2 + 2y + 1)$$

$$\implies \frac{1}{y^2 + 2y + 1}dy = \frac{-4x}{x^2 + 2}dx$$

Observe here that dividing by $y^2 + 2y + 1$ poses an issue if it is 0, so let's check that

$$y^2 + 2x + 1 = (y + 1)^2 = 0 \Leftrightarrow y = -1.$$

We can plug in $y \equiv -1$ and verify that it is indeed a solution. Notice that $y \equiv -1$ satisfies the initial value $y(1) = -1$. This means that the solution to the IVP is $y \equiv -1$. This solution is valid for all $x$.

(If you were to try to solve the equation by separation of variables first, you'll see that you cannot solve for $C$, which should lead you to check the constant solution.) □

# Section 2.4.

**Exercise 7.** *Solve the initial value problem*

$$y' - 2y = xy^3, \;\; y(0) = 2\sqrt{2}.$$

*Solution.* This is a Bernoulli equation with $n = 3$, so we make the substitution $u = y^{1-3} = y^{-2}$. This substitution turns the equation into

$$\frac{1}{-2}u' - 2u = x.$$

Rearrange this into a linear equation

$$u' + 4u = -2x$$

and use the usual techniques to solve it:

$$\mu(x) = e^{\int^x 4ds} = e^{4x}$$

so

$$u = e^{-4x}\int (e^{4x})(-2x)dx = e^{-4x}\int -2xe^{4x}dx$$

Use integration by parts with $u = -2x$ and $dv = e^{4x}dx$ to get

$$\begin{aligned} u &= e^{-4x}\left(-\frac{1}{2}xe^{4x} - \int \left(\frac{1}{4}e^{4x}\right)(-2)dx\right) \\ &= e^{-4x}\left(-\frac{1}{2}xe^{4x} + \frac{1}{2}\int e^{4x}dx\right) \\ &= e^{-4x}\left(-\frac{1}{2}xe^{4x} + \frac{1}{8}e^{4x} + C\right) \\ &= -\frac{1}{2}x + \frac{1}{8} + Ce^{-4x} \end{aligned}$$

Now, $u = y^{-2}$ so

$$y^{-2} = -\frac{1}{2}x + \frac{1}{8} + Ce^{-4x}.$$

Use the initial value to get

$$\begin{aligned} (2\sqrt{2})^{-2} &= -\frac{1}{2}(0) + \frac{1}{8} + Ce^{-4(0)} \\ \frac{1}{8} &= \frac{1}{8} + C \end{aligned}$$

Thus we have that $C = 0$ and the solution is

$$y^{-2} = -\frac{1}{2}x + \frac{1}{8}.$$

or

$$\begin{aligned} y^2 &= \frac{1}{-\frac{1}{2}x + \frac{1}{8}} \\ &= \frac{8}{1 - 4x} \\ y &= \frac{2\sqrt{2}}{\sqrt{1 - 4x}} \end{aligned}$$

□

**Exercise 19.** *Solve the equation explicitly and plot a direction field and some integral curves on the indicated rectangular region*

$$x^2y' = xy + x^2 + y^2;\ \{-8 \le x \le 8, -2 \le y \le 8\}.$$

*Solution.* Begin by dividing the equation by $x^2$:

$$y' = \frac{y}{x} + 1 + \frac{y^2}{x^2} = \frac{y}{x} + 1 + \left(\frac{y}{x}\right)^2.$$

From here it is easy to see that the equation is homogeneous. Letting $u = \frac{y}{x}$, we get that $y' = u'x + u$ and so the differential equation becomes

$$u'x + u = u + 1 + u^2$$

and simplified

$$u'x = 1 + u^2.$$

Separate this to get

$$\frac{1}{1+u^2}du = \frac{1}{x}dx.$$

Integrate and we have

$$\arctan u = \ln|x| + C.$$

Solve for $u$

$$u = \tan(\ln|x| + C)$$

and replace $u$ with $\frac{y}{x}$

$$\frac{y}{x} = \tan(\ln|x| + C)$$

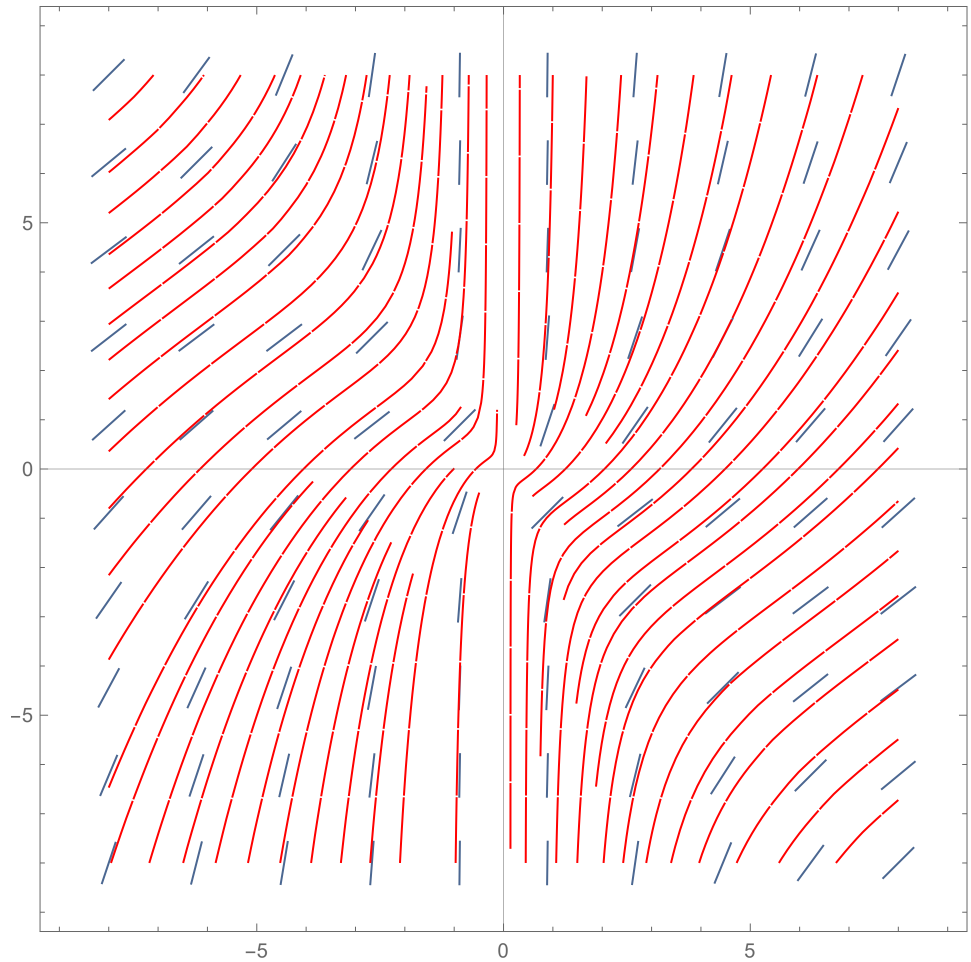
giving a final explicit solution of

$$y = x\tan(\ln|x| + C).$$

The *Mathematica* code

```
Show[VectorPlot[{x^2, x y + x^2 + y^2}, {x, -8, 8}, {y, -8, 8},
  VectorStyle -> Arrowheads[0], Axes -> True,
  VectorScale -> {Tiny, Automatic, None},
  VectorPoints -> 10],
 StreamPlot[{x^2, x y + x^2 + y^2}, {x, -8, 8}, {y, -8, 8},
  StreamStyle -> {Red, Arrowheads[0]}]
 ]
```

will generate the direction field and integral curves


5
0
−5
−5
0
5

□

# 3. Homework 3 (2/21 & 2/23)

## Section 2.3.

**Exercise 1.** *Find all* $(x_0, y_0)$ *for which Theorem 2.3.1 (the Existence and Uniqueness Theorem) implies that the initial value problem*

$$y' = \frac{x^2 + y^2}{\sin x}, \ y(x_0) = y_0$$

*has (a) a solution; (b) a unique solution on some open interval containing* $x_0$.

*Solution.*

(a) $f(x, y) = \dfrac{x^2 + y^2}{\sin x}$ is continuous as long as $x \neq n\pi$, so solutions exist for initial values $(x_0, y_0)$ with $x_0 \neq n\pi$ for any integer $n$.

(b) $f_y(x, y) = \dfrac{2y}{\sin x}$ is continuous as long as $x \neq n\pi$, so solutions are unique for initial values $(x_0, y_0)$ with $x_0 \neq n\pi$.

□

## Section 2.5.

**Exercise 9.** *Determine if the equation is exact and solve if it is*

$$(3x^2 + 2xy + 4y^2)dx + (x^2 + 8xy + 18y)dy = 0.$$

*Solution.*

$$\begin{aligned} \frac{\partial}{\partial y}(3x^2 + 2xy + 4y^2) &= 2x + 8y \\ \frac{\partial}{\partial x}(x^2 + 8xy + 18y) &= 2x + 8y \end{aligned}$$

So the equation is exact. Let's integrate the first piece with respect to $x$ to find $f$:

$$f = \int (3x^2 + 2xy + 4y^2)dx = x^3 + x^2y + 4xy^2 + g(y).$$

Differentiate with respect to $y$ and set it equal to the second piece

$$\frac{\partial f}{\partial y} = x^2 + 8xy + g'(y).$$

This gives that

$$g'(y) = 18y \quad \Longrightarrow \quad g(y) = 9y^2 + C.$$

Thus the solution is

$$f = x^3 + x^2y + 4xy^2 + 9y^2 = C.$$

□

# Section 2.6.

**Exercise 11.** *Find an integrating factor that is a function of only one variable and solve the differential equation*

$$(12x^3y + 24x^2y^2)dx + (9x^4 + 32x^3y + 4y)dy = 0.$$

*Solution.* Let's try finding an integrating factor of the form

$$\mu(x, y) = x^m y^n.$$

Multiply the equation by this

$$(12x^{m+3}y^{n+1} + 24x^{m+2}y^{n+2})dx + (9x^{m+4}y^n + 32x^{m+3}y^{n+1} + 4x^m y^{n+1})dy = 0$$

and now differentiate to check the exactness condition ($M_y = N_x$)

$$\begin{aligned} M_y &= 12(n+1)x^{m+3}y^n + 24(n+2)x^{m+2}y^{n+1} \\ N_x &= 9(m+4)x^{m+3}y^n + 32(m+3)x^{m+2}y^{n+1} + 4mx^{m-1}y^{n+1} \end{aligned}$$

Comparing coefficients of the like terms, we get

$$\begin{aligned} 12(n+1) &= 9(m+4) \\ 24(n+2) &= 32(m+3) \\ 0 &= 4m \end{aligned}$$

The last equation gives $m = 0$, so plugging this in the first two gives

$$\begin{aligned} 12n + 12 &= 36 \\ 24n + 48 &= 96 \end{aligned}$$

and we find that $n = 2$ is the solution in both cases. Thus the integrating factor is $\mu = y^2$ (which is a function of one variable). When we multiply the original equation by this we get

$$(12x^3y^3 + 24x^2y^4)dx + (9x^4y^2 + 32x^3y^3 + 4y^3)dy = 0$$

which is now exact. Proceed as normal to solve it: Integrate the first part with respect to $x$

$$f = \int (12x^3y^3 + 24x^2y^4)dx = 3x^4y^3 + 8x^3y^4 + g(y).$$

Now differentiate with respect to $y$:

$$\frac{\partial f}{\partial y} = 9x^4y^2 + 32x^3y^3 + g'(y)$$

and comparing it to $N$ we have that

$$g'(y) = 4y^3 \quad \Longrightarrow \quad g(y) = y^4 + C.$$

So the solution is

$$f = 3x^4y^3 + 8x^3y^4 + y^4 = C.$$

□

# Handout.

**Exercise 5.** *Solve the initial value problem*

$$y' = -\frac{y}{2} + t, \; y(0) = 0$$

*by letting $\phi_0(t) = 0$ and using the method of successive approximations.*

(a) *Determine $\phi_n(t)$ for an arbitrary value of $n$.*
(b) *Plot $\phi_n(t)$ for $n = 1, ..., 4$. Observe whether the iterate appear to be converging.*
(c) *Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.*
(d) *Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, ..., 4$. For each of $\phi_1(t), ..., \phi_4(t)$, estimate the interval in which it is a reasonable good approximation to the actual solution.*

*Solution.*

(a)

$$\begin{aligned}
\phi_0 &= 0 \\
\phi_1 &= \int_0^t \left(-\frac{1}{2}\phi_0 + s\right) ds \\
&= \int_0^t (s)\, ds \\
&= \frac{1}{2}t^2 \\
\phi_2 &= \int_0^t \left(-\frac{1}{2}\phi_1 + s\right) ds \\
&= \int_0^t \left(-\frac{1}{2^2}s^2 + s\right) ds \\
&= -\frac{1}{2^2 3}t^3 + \frac{1}{2}t^2 \\
\phi_3 &= \int_0^t \left(-\frac{1}{2}\phi_2 + s\right) ds \\
&= \int_0^t \left(\frac{1}{2^3 3}s^3 - \frac{1}{2^2}s^2 + s\right) ds \\
&= \frac{1}{2^3 3 \cdot 4}t^4 - \frac{1}{2^2 3}t^3 + \frac{1}{2}t^2 \\
\phi_4 &= \int_0^t \left(-\frac{1}{2}\phi_3 + s\right) ds \\
&= \int_0^t \left(-\frac{1}{2^4 3 \cdot 4}s^4 + \frac{1}{2^3 3}s^3 - \frac{1}{2^2}s^2 + s\right) ds \\
&= -\frac{1}{2^4 3 \cdot 4 \cdot 5}t^5 + \frac{1}{2^3 3 \cdot 4}t^4 - \frac{1}{2^2 3}t^3 + \frac{1}{2}t^2
\end{aligned}$$

Following the pattern, it's looking like

$$\phi_n = \sum_{k=1}^{n} \frac{(-1)^{n+1}}{2^{n-1}} \frac{t^{n+1}}{(n+1)!}$$

Run this through the algorithm one more time to be sure

$$\begin{aligned} \phi_{n+1} &= \int_0^t \left( -\frac{1}{2}\phi_n + s \right) ds \\ &= \int_0^t \left( \sum_{k=1}^{n} \frac{(-1)^{n+2}}{2^n} \frac{s^{n+1}}{(n+1)!} + s \right) ds \\ &= \sum_{k=1}^{n} \frac{(-1)^{n+2}}{2^n} \frac{t^{n+2}}{(n+2)!} + \frac{1}{2}t^2 \\ &= \sum_{k=2}^{n+1} \frac{(-1)^{n+1}}{2^{n-1}} \frac{t^{n+1}}{(n+1)!} + \frac{1}{2}t^2 \\ &= \sum_{k=1}^{n+1} \frac{(-1)^{n+1}}{2^{n-1}} \frac{t^{n+1}}{(n+1)!} \end{aligned}$$

This verifies that the general form is

$$\phi_n = \sum_{k=1}^{n} \frac{(-1)^{n+1}}{2^{n-1}} \frac{t^{n+1}}{(n+1)!}$$

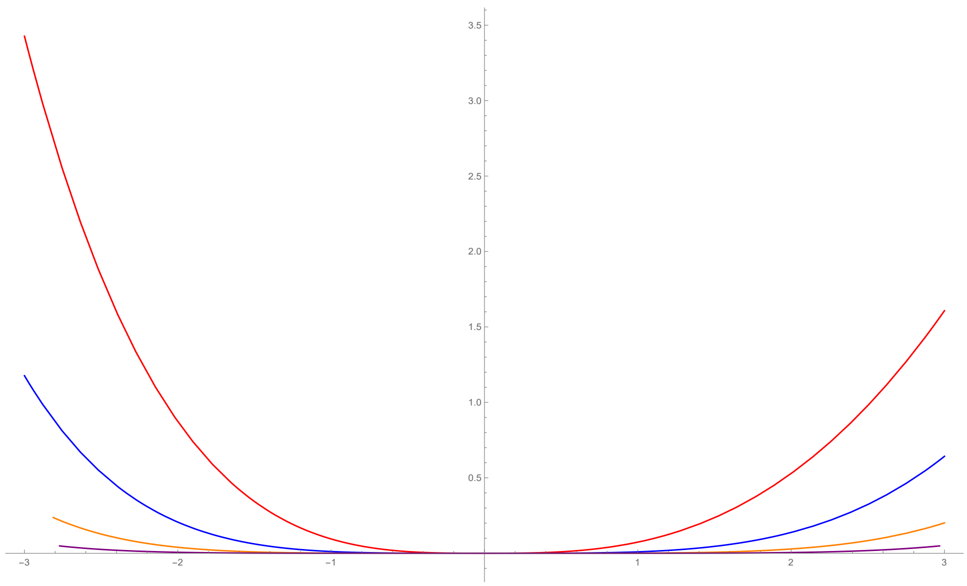
(b) $\phi_1$ is in red, $\phi_2$ is in blue, $\phi_3$ is in orange, and $\phi_4$ is in purple.


The iterates do seem to be converging toward a function.

(c)

$$\begin{aligned}
\phi &= \lim_{n\to\infty}\phi_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{n+1}}{2^{n-1}}\frac{t^{n+1}}{(n+1)!} \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{2^{n-1}}\frac{t^{n+1}}{(n+1)!} = \sum_{k=1}^{\infty}4\frac{(-1)^{n+1}}{2^{n+1}}\frac{t^{n+1}}{(n+1)!} \\
&= \sum_{k=2}^{\infty}4\frac{(-1)^{n}}{2^{n}}\frac{t^{n}}{n!} = 4\sum_{k=2}^{\infty}\frac{1}{n!}\frac{(-1)^n t^n}{2^n} \\
&= 4\left(\sum_{k=2}^{\infty}\frac{1}{n!}\left(\frac{-t}{2}\right)^n\right) = 4\left(\sum_{k=0}^{\infty}\frac{1}{n!}\left(\frac{-t}{2}\right)^n - \left(-\frac{t}{2}+1\right)\right) \\
&= 4e^{-t/2} + 2t - 4
\end{aligned}$$

(d) The colors below correspond to the same $n$ values




$\phi_1$ is a good approximation on $(-0.4, 0.4)$, $\phi_2$ is a good approximation on $(-0.9, 0.9)$, $\phi_3$ is a good approximation on $(-1.4, 1.6)$, and $\phi_2$ is a good approximation on $(-2, 2.2)$,

□

# 4. Homework 4 (2/28 & 3/2)

## Section 3.1.

**Exercise 15.** *Use Euler's method with the indicated step sizes to find approximate values of the solution of the given initial value problem at* 11 *equally spaces points. Also graph the approximate solution for each of the step sizes.*

$$y' + 2xy = x^2, \ y(0) = 3.$$

*Solution.* The chart of approximations: □

## Section 3.2.

**Exercise 15.** *Use the Improved Euler's method with the indicated step sizes to find approximate values of the solution of the given initial value problem at* 11 *equally spaces points. Also graph the approximate solution for each of the step sizes.*

$$y' + 2xy = x^2, \ y(0) = 3.$$

*Solution.* □

## Section 3.3.

**Exercise 15.** *Use the Runge-Kutta method with the indicated step sizes to find approximate values of the solution of the given initial value problem at* 11 *equally spaces points. Also graph the approximate solution for each of the step sizes.*

$$y' + 2xy = x^2, \ y(0) = 3.$$

*Solution.* □

**Section 4.5.**

**Exercise 25.** *Find the orthogonal trajectories of the family*

$$x^2 + 2y^2 = c^2$$

*Solution.* First find $y'$:

$$2x + 4yy' = 0$$

so

$$y' = \frac{-2x}{4y} = -\frac{x}{2y}.$$

The orthogonal slope is then

$$y' = \frac{2y}{x}$$

and we solve this differential equation by separation:

$$\frac{1}{y}y' = \frac{2}{x}$$

integrate

$$\ln|y| = 2\ln|x| + k = \ln x^2 + k$$

exponentiate

$$|y| = e^{\ln x^2 + k} = ke^{\ln x^2} = kx^2$$

absorb the absolute value into the constant

$$y = kx^2.$$

Thus the family $y = kx^2$ is the orthogonal family of curves. □

**Exercise 28.** *Find the orthogonal trajectories of the family*

$$xye^{x^2} = c$$

*Solution.* Begin by differentiating to find $y'$:

$$ye^{x^2} + xy'e^{x^2} + 2x^2ye^{x^2} = 0.$$

Divide by $e^{x^2}$ to simplify:

$$y + xy' + 2x^2y = 0$$

and solve for $y'$:

$$y' = -\frac{(1+2x^2)y}{x}.$$

Now, get the orthogonal slope

$$y' = \frac{x}{(1+2x^2)y}$$

and solve this by separation

$$yy' = \frac{x}{1+2x^2}$$

and integrate both sides

$$\frac{1}{2}y^2 = \frac{1}{4}\ln(1+2x^2) + C.$$

Thus the family of orthogonal curves is

$$y^2 = \frac{1}{2}\ln(1+2x^2) + C.$$

□

# 5. Homework 5 (3/7 & 3/9)

## Section 5.1.

**Exercise 5.** *Compute the Wronskian of the given sets of functions*

(b) $\{e^x, e^x \sin x\}$
(d) $\{x^{1/2}, x^{-1/3}\}$

*Solution.*

(b)

$$\begin{aligned} W(x; e^x, e^x \sin x) &= \begin{vmatrix} e^x & e^x \sin x \\ e^x & e^x \sin x + e^x \cos x \end{vmatrix} \\ &= e^{2x}(\sin x + \cos x) - e^{2x} \sin x = e^{2x} \cos x \end{aligned}$$

(d)

$$\begin{aligned} W(x; x^{1/2}, x^{-1/3}) &= \begin{vmatrix} x^{1/2} & x^{-1/3} \\ \frac{1}{2}x^{-1/2} & -\frac{1}{3}x^{-4/3} \end{vmatrix} \\ &= -\frac{1}{3}x^{-5/6} - \frac{1}{2}x^{-5/6} = -\frac{5}{6}x^{-5/6} \end{aligned}$$

□

**Exercise 8.** *Find the Wronskian of a given set* $\{y_1, y_2, \}$ *of solutions of*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

*given that* $W(1) = 1$.

*Solution.* Use Abel's Formula to do this, but first we need to put the differential equation into the proper form by dividing by $x^2$:

$$y'' + \frac{1}{x}y' + \frac{x^2 - \nu^2}{x^2}y = 0,$$

then $q(x) = \dfrac{1}{x}$ and using Abel's formula we have

$$W(x) = W(1)e^{-\int_1^x \frac{1}{s}ds} = (1)e^{-\ln|x|} = \frac{1}{x}.$$

□

## Section 5.2.

**Exercise 1.** *Find the general solution of*

$$y'' + 5y' - 6y = 0.$$

*Solution.* The characteristic equation is

$$r^2 + 5r - 6 = (r + 6)(r - 1) = 0$$

so the roots are $r = -6, 1$. Thus the general solution is

$$y = c_1e^{-6x} + c_2e^x.$$

□

**Exercise 5.** *Find the general solution of*

$$y'' + 2y' + 10y = 0.$$

*Solution.* The characteristic equation is

$$r^2 + 2r + 10 = 0$$

so the roots are

$$r = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i.$$

Thus the general solution is

$$y = c_1 e^{-x} \cos 3x + c_2 e^{-x} \sin 3x.$$

□

**Exercise 17.** *Find the solution of the initial value problem*

$$4y'' - 12y' + 9y = 0, y(0) = 3, y'(0) = \frac{5}{2}.$$

*Solution.* The characteristic equation is

$$4r^2 - 12r + 9 = 4\left(r^2 - 3r + \frac{9}{4}\right) = 4\left(r - \frac{3}{2}\right)^2 = 0$$

so there is a repeated root $r = \frac{3}{2}$. Thus the general solution is

$$y = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}.$$

Use the initial value $y(0) = 3$ to get

$$y(0) = c_1 e^0 + c_2(0)e^0 = c_1 = 3.$$

Plug this in and take the derivative so we can use the second initial value:

$$y' = \frac{9}{2}e^{\frac{3}{2}x} + c_2\left(e^{\frac{3}{2}x} + \frac{3}{2}xe^{\frac{3}{2}x}\right)$$

and applying the initial value $y'(0) = \frac{5}{2}$

$$y'(0) = \frac{9}{2}e^0 + c_2\left(e^0 + \frac{3}{2}(0)e^0\right) = \frac{9}{2} + c_2 = \frac{5}{2}$$

so that $c_2 = -2$. Thus the solution to the IVP is

$$y = 3e^{\frac{3}{2}x} - 2xe^{\frac{3}{2}x}.$$

□

# 6. Homework 6 (3/16)

## Section 5.3.

**Exercise 1.** *Find the general solution of*

$$y'' + 5y' - 6y = 22 + 18x - 18x^2.$$

*Solution.* Begin by finding the roots of the characteristic polynomial. The characteristic polynomial is

$$r^2 + 5r - 6 = (r + 6)(r - 1) = 0$$

so the roots are

$$r = -6, 1.$$

Since 0 is not a root, we make the guess that $y_p = A_0 + A_1 x + A_2 x^2$. Then

$$\begin{aligned} y_p' &= A_1 + 2A_2 x \\ y_p'' &= 2A_2 \end{aligned}$$

and plugging these in gives

$$\begin{aligned} y_p'' + 5y_p' - 6y_p &= 2A_2 + 5A_1 + 10A_2 x - 6A_0 - 6A_1 x - 6A_2 x^2 \\ &= (2A_2 + 5A_1 - 6A_0) + (10A_2 - 6A_1)x + (-6A_2)x^2 = 22 + 18x - 18x^2 \end{aligned}$$

This gives us the system of equations

$$\begin{aligned} 2A_2 + 5A_1 - 6A_0 &= 22 \\ 10A_2 - 6A_1 &= 18 \\ -6A_2 &= -18 \end{aligned}$$

The third equation gives $A_2 = 3$; plugging this in the second equation gives $30 - 6A_1 = 18$, so then $A_1 = 2$; and plugging both of these into the first equation gives $6 + 10 - 6A_0 = 22$ which means $A_0 = -1$. Therefore, the particular solution is

$$y_p = -1 + 2x + 3x^2$$

and the general solution is

$$y_G = c_1 e^{-6x} + c_2 e^x - 1 + 2x + 3x^2.$$

□

**Exercise 16.** *Find the general solution of*

$$y'' + 5y' - 6y = 6e^{3x}.$$

*Solution.* From the previous problem we know that the roots of the characteristic polynomial are

$$r = -6, 1$$

so since 3 isn't a root, our guess will be

$$y_p = Ae^{3x}.$$

Then

$$y_p' = 3Ae^{3x} \quad \text{and} \quad y_p'' = 9Ae^{3x}.$$

Plugging this in gives

$$9Ae^{3x} + 15Ae^{3x} - 6Ae^{3x} = 18Ae^{3x} = 6e^{3x}.$$

Thus $18A = 6$ which gives $A = \frac{1}{3}$. So the particular solution is

$$y_p = \frac{1}{3}e^{3x}$$

and the general solution is

$$y_G = c_1e^{-6x} + c_2e^x + \frac{1}{3}e^{3x}.$$

□

**Exercise 33.** *Find the general solution of*

$$y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}.$$

*Solution.* Since the left side is the same as the previous two problems and the function $f(x) = 22 + 18x - 18x^2 + 6e^{3x}$ is just the sum of the right sides of the previous two problems, we can just superimpose the particular solutions to get the general solution here to be

$$y_G = c_1e^{-6x} + c_2e^x - 1 + 2x + 3x^2 + 6e^{3x}.$$

□

# Section 5.4.

**Exercise 21.** *Solve the initial value problem*

$$y'' + 3y' - 4y = e^{2x}(7 + 6x), \quad y(0) = 2, \quad y'(0) = 8.$$

*Solution.* The characteristic polynomial is

$$r^2 + 3r - 4 = (r + 4)(r - 1) = 0$$

so the roots are

$$r = -4, 1.$$

Since 2 is not a root, we make the guess

$$y_p = e^{2x}(A + Bx).$$

The derivatives are

$$y_p' = 2e^{2x}(A + Bx) + Be^{2x}$$

and

$$y_p'' = 4e^{2x}(A + Bx) + 4Be^{2x}.$$

Plugging these in gives

$$\begin{aligned} y_p'' + 3y_p' - 4y_p &= 4e^{2x}(A + Bx) + 4Be^{2x} + 6e^{2x}(A + Bx) + 3Be^{2x} - 4e^{2x}(A + Bx) \\ &= e^{2x}\left((4A + 4B + 6A + 3B - 4A) + (4B + 6B - 4B)x\right) \\ &= e^{2x}\left((6A + 7B) + (6B)x\right) = e^{2x}(7 + 6x) \end{aligned}$$

Thus

$$\begin{aligned} 6A + 7B &= 7 \\ 6B &= 6 \end{aligned}$$

so $B = 1$ and $A = 0$. Thus the particular solution is

$$y_p = xe^{2x}$$

and the general solution is

$$y_G = c_1 e^{-4x} + c_2 e^x + xe^{2x}.$$

The derivative is

$$y' = -4c_1 e^{-4x} + c_2 e^x + e^{2x} + 2xe^{2x}.$$

Using the initial value $y(0) = 2$ gives

$$y(0) = c_1 + c_2 = 2$$

and using $y'(0) = 8$ gives

$$y'(0) = -4c_1 + c_2 + 1 = 8.$$

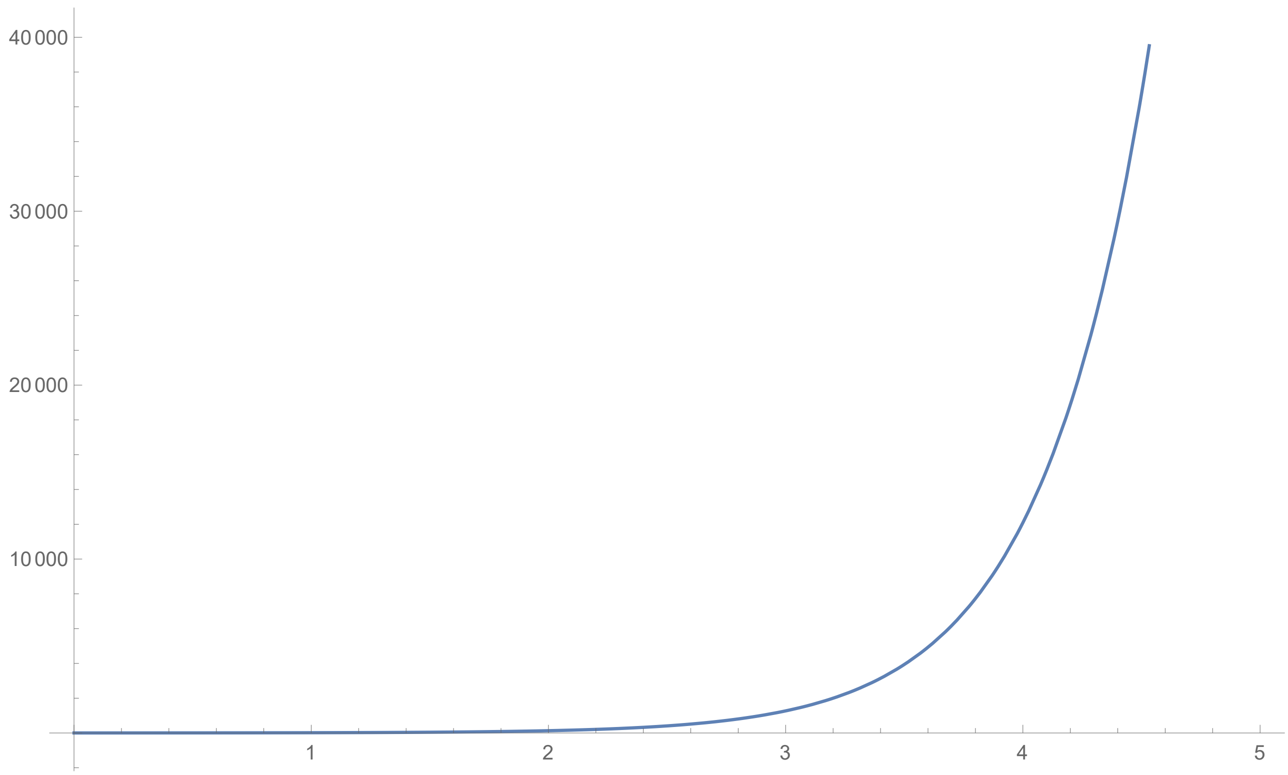
So we have the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ -4c_1 + c_2 &= 7 \end{aligned}$$

Solving this we can find that $c_1 = -1$ and $c_2 = 3$ so that the solution to the initial value problem is

$$y = -e^{-4x} + 3e^x + xe^{2x}.$$

The graph of the solution on $[0, 5]$ is




□

# 7. Homework 7 (3/21 & 3/23)

## Section 5.5.

**Exercise 5.** *Find a particular solution of*

$$y'' - y' + y = e^x(2+x)\sin x.$$

*Solution.* Let's first make the substitution $y = e^x u$. Then

$$\begin{aligned} y'' - y' + y &= (e^x u + 2e^x u' + e^x u'') - (e^x u + e^x u') + (e^x u) \\ &= e^x(u + 2u' + u'' - u - u' + u) \\ &= e^x(u'' + u' + u) = e^x(2+x)\sin x \end{aligned}$$

So the differential equation in $u$ is

$$u'' + u' + u = (2+x)\sin x$$

The characteristic polynomial of this equation is

$$r^2 + r + 1 = 0$$

which has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Since $i$ is not a root, we make the guess

$$\begin{aligned} u_p &= (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x \\ u_p' &= (A_1 + B_0 + B_1 x)\cos x + (-A_0 + B_1 - A_1 x)\sin x \\ u_p'' &= (-A_0 + 2B_1 - A_1 x)\cos x + (-2A_1 - B_0 - B_1 x)\sin x \end{aligned}$$

Which we now plug into the differential equation

$$\begin{aligned} u_p'' + u_p' + u_p &= (-A_0 + 2B_1 - A_1 x)\cos x + (-2A_1 - B_0 - B_1 x)\sin x \\ &\quad + (A_1 + B_0 + B_1 x)\cos x + (-A_0 + B_1 - A_1 x)\sin x \\ &\quad + (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x \\ &= (A_1 + B_0 + 2B_1 + B_1 x)\cos x + (-A_0 - 2A_1 + B_1 - A_1 x)\sin x \\ &= (2+x)\sin x \end{aligned}$$

Comparing coefficients gives us the system of equations

$$\begin{cases} A_1 + B_0 + 2B_1 = 0 \\ B_1 = 0 \\ -A_0 - 2A_1 + B_1 = 2 \\ -A_1 = 1 \end{cases}$$

Solving this system gives

$$A_0 = 0, A_1 = -1, B_0 = 1, B_1 = 0,$$

so

$$u_p = -x\cos x + \sin x$$

and the particular solution to the original equation is

$$y_p = e^x(-x\cos x + \sin x).$$

□

**Exercise 23.** *Solve the initial value problem*

$$y'' - 2y' + 2y = -e^x(6\cos x + 4\sin x),\ \ y(0) = 1,\ \ y'(0) = 4.$$

*Solution.* The characteristic polynomial for the homogeneous equation is

$$r^2 - 2r + 2 = 0$$

which has roots

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

so the homogeneous solution is

$$y_H = e^x(c_1\cos x + c_2 \sin x).$$

Now, let $y_p = e^x u$ and plug into the differential equation

$$\begin{aligned} y'' - 2y' + 2y &= (e^x u + 2e^x u' + e^x u'') - 2(e^x u + e^x u') + 2(e^x u) \\ &= e^x(u + 2u' + u'' - 2u - 2u' + 2u) \\ &= e^x(u'' + u) = -e^x(6\cos x + 4\sin x) \end{aligned}$$

Thus we have

$$u'' + u = -(6\cos x + 4\sin x).$$

The characteristic polynomial here is

$$r^2 + 1 = 0$$

which has roots $r = \pm i$, so our guess needs to be

$$\begin{aligned} u_p &= x(A\cos x + B\sin x) \\ u_p'' &= 2(-A\sin x + B\cos x) + x(-A\cos x - B\sin x) \\ &= (2B - Ax)\cos x + (-2A - Bx)\sin x \end{aligned}$$

Plugging this into the differential equation gives

$$\begin{aligned} u_p'' + u_p &= (2B - Ax)\cos x + (-2A - Bx)\sin x + x(A\cos x + B\sin x) \\ &= 2B\cos x - 2A\sin x = -(6\cos x + 4\sin x) \end{aligned}$$

from which we see that $B = -3$ and $A = 2$, so that

$$u_p = 2\cos x - 3\sin x.$$

Thus the general solution to the original equation is

$$y_G = e^x(c_1\cos x + c_2\sin x + 2x\cos x - 3x\sin x).$$

Now, we use the initial values. Using $y(0) = 1$ we get

$$y(0) = c_1 = 1.$$

Now take the derivative of the solution:

$$y' = e^x(\cos x + c_2\sin x + 2x\cos x - 3x\sin x) + e^x(-\sin x + c_2\cos x + 2\cos x - 2x\sin x - 3\sin x - 3x\cos x)$$

and use the initial value $y'(0) = 4$:

$$y'(0) = 1 + c_2 + 2 = 4 \quad \Longrightarrow \quad c_2 = 1.$$

So the solution to the IVP is

$$y = e^x(c_1\cos x + c_2\sin x + 2x\cos x - 3x\sin x).$$

□

## Section 5.6.

**Exercise 1.** *Find the general solution of*

$$(2x+1)y'' - 2y' - (2x+3)y = (2x+1)^2$$

*given that $y_1 = e^{-x}$ is a homogeneous equation. As a byproduct, find a fundamental set of solutions of the homogeneous equation.*

*Solution.* □

## Section 7.4.

**Exercise 7.** *Solve the Cauchy-Euler equation*

$$x^2y'' + 3xy' - 3y = 0, \; x > 0.$$

*Solution.* The indicial equation for this is

$$m(m-1) + 3m - 3 = m^2 + 2m - 3 = (m+3)(m-1) = 0$$

so the roots are $m = -3, 1$. Thus the solution is

$$y = c_1x^{-3} + c_2x.$$

□

# 8. Homework 8 (4/4 & 4/6)

## Section 5.7.

**Exercise 1.** *Use variation of parameters to find a particular solution of*

$$y'' + 9y = \tan 3x.$$

*Solution.* The characteristic equation is $r^2 + 9 = 0$ which has roots $r = \pm 3i$ so the fundamental solutions are

$$y_1 = \cos 3x \quad \textit{and} \quad y_2 = \sin 3x.$$

Now set up the Variation of Parameters system:

$$\begin{array}{rcrcll} u_1' \cos 3x & + & u_2' \sin 3x & = & 0 & \textcircled{1} \\ -2u_1' \sin 3x & + & 2u_2' \cos 3x & = & \tan 3x & \textcircled{2} \end{array}$$

Taking $3\sin 3x \textcircled{1} + \cos 3x \textcircled{2}$ gives

$$3u_2' \sin^2 3x + 3u_2' \cos^2 3x = \sin 3x$$

which we solve for $u_2'$ to get

$$u_2' = \frac{1}{3}\sin 3x \quad \Longrightarrow \quad u_2 = -\frac{1}{9}\cos 3x.$$

Plugging $u_2' = \frac{1}{3}\sin 3x$ in for $u_2'$ in $\textcircled{1}$ and solving for $u_1'$ gives

$$u_1' = -\frac{1}{3}\frac{\sin^2 3x}{\cos 3x} = -\frac{1}{3}\frac{1-\cos^2 3x}{\cos 3x} = -\frac{1}{3}\left(\sec 3x - \cos 3x\right)$$

and integrating gives

$$u_1 = -\frac{1}{9}\left(\ln|\sec 3x + \tan 3x| - \sin 3x\right).$$

Finally, combining everything together, we get

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= -\frac{1}{9}\left(\ln|\sec 3x + \tan 3x| - \sin 3x\right)\cos 3x - \frac{1}{9}\cos 3x \sin 3x \\ &= -\frac{\cos 3x \ln|\sec 3x + \tan 3x|}{9} \end{aligned}$$

□

**Exercise 7.** *Use variation of parameters to find a particular solution of*

$$x^2 y'' + xy' - y = 2x^2 + 2$$

*given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are homogeneous solutions.*

*Solution.* Set up the Variation of Parameters system:

$$\begin{array}{rcrcll} u_1' x & + & u_2' x^{-1} & = & 0 & \textcircled{1} \\ u_1' & + & -u_2' x^{-2} & = & 2 + 2x^{-2} & \textcircled{2} \end{array}$$

Take $\textcircled{1} + x\textcircled{2}$ to get

$$2xu_1' = 2x + 2x^{-1}$$

and solve for $u_1'$ to get

$$u_1' = 1 + x^{-2} \quad \Longrightarrow \quad u_1 = x - x^{-1}.$$

Plug $u_1'$ into ① and solve for $u_2'$ to get

$$u_2' = -x^2 - 1 \quad \Longrightarrow \quad u_2 = -\frac{1}{3}x^3 - x.$$

Bringing this together, we get

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (x - x^{-1})x + \left(-\frac{1}{3}x^3 - x\right) \\ &= \frac{2}{3}x^2 - 2 = \frac{2(x^2-3)}{3} \end{aligned}$$

□

**Exercise 19.** *Use variation of parameters to find a particular solution of*

$$(\sin x)y'' + (2\sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}$$

*given that $y_1 = e^{-x}$ and $y_2 = e^{-x}\cos x$ are homogeneous solutions.*

*Solution.* Set up the Variation of Parameters system:

$$\begin{array}{rcrcll} u_1' e^{-x} & + & u_2' e^{-x}\cos x & = & 0 & \textcircled{1} \\ -u_1' e^{-x} & + & u_2'(-e^{-x}\cos x - e^{-x}\sin x) & = & e^{-x}\csc x & \textcircled{2} \end{array}$$

Take ① + ② to get

$$-u_2' e^{-x}\sin x = e^{-x}\csc x$$

and solve for $u_2'$ to get

$$u_2' = -\csc^2 x \quad \Longrightarrow \quad u_2 = \cot x.$$

Plug $u_2'$ into ① and solve for $u_1'$ to get

$$u_1' = -\frac{\cos^2 x}{\sin x} = -(\csc x + \sin x).$$

Integrating gives us

$$u_1 = \ln|\csc x + \cot x| + \cos x.$$

Bringing this together, we get

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (\ln|\csc x + \cot x| + \cos x)e^{-x} + (\cot x)\, e^{-x}\cos x \\ &= e^{-x}\left(\ln|\csc x + \cot x| + \cos x + \cos x \cot x\right) \end{aligned}$$

Since $e^{-x}\cos x$ is a homogeneous solution, we can drop it from the particular solution (not required, but it is good practice) to get

$$y_p = e^{-x}\left(\ln|\csc x + \cot x| + \cos x \cot x\right).$$

□

# Section 6.1.

**Exercise 3.** *A spring with natural length* .5 *m has length* 50.5 *cm with a mass of* 2 *g suspended from it. The mass is initially displaced* 1.5 *cm below equilibrium and released with zero velocity. Find its displacement for* $t > 0$.

*Solution.* We have $\Delta l = .5$, $m = 2$, $y(0) = -1.5$, and $y'(0) = 0$. We need to set up $y'' + \omega_0^2 y = 0$. Since $mg = k\Delta l$, we have

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta l}} = \sqrt{\frac{980}{.5}} = \sqrt{1960} = 14\sqrt{10}$$

and so the differential equation is

$$y'' + 1960y = 0.$$

The solution to this is

$$y = c_1 \cos(14\sqrt{10}t) + c_2 \sin(14\sqrt{10}t).$$

First we use the initial value $y(0) = -1.5$:

$$y(0) = c_1 = -1.5.$$

Now take the derivative

$$y' = 21\sqrt{10}\sin(14\sqrt{10}t) + c_2 14\sqrt{10}\cos(14\sqrt{10}t)$$

and plug in the initial value

$$y'(0) = 14\sqrt{10}c_2 = 0$$

so that $c_2 = 0$. Thus the displacement function is

$$y = -1.5\cos(14\sqrt{10}t).$$

□

**Exercise 15.** *A* $6 - lb$ *weight stretches a spring* 6 *inches in equilibrium. Suppose that an external force* $F(t) = \frac{3}{16}\sin\omega t + \frac{3}{8}\cos\omega t$ *lb is applied to the weight. For what value of* $\omega$ *will the displacement be unbounded? Find the displacement if* $\omega$ *has this value. Assume that the motion starts from equilibrium with zero initial value.*

*Solution.* We have $m = 6$, $\Delta l = 0.5\ ft$, $y(0) = 0$, and $y'(0) = 0$. The displacement is unbounded if $\omega = \omega_0 = \sqrt{\frac{k}{m}}$, i.e., the frequency of the driving force equals the natural frequency of the spring-mass system. So set

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta l}} = \sqrt{\frac{32}{0.5}} = \sqrt{64} = 8.$$

Then the differential equation we want to solve is

$$6y'' + ky = \frac{3}{16}\sin 8t + \frac{3}{8}\cos 8t$$

or

$$y'' + 64y = \frac{1}{32}\sin 8t + \frac{1}{16}\cos 8t.$$

The characteristic polynomial is $r^2 + 64 = 0$ so the roots are $r = \pm 8i$, thus the homogeneous solution is

$$y_H = c_1 \cos 8t + c_2 \sin 8t.$$

Since $\pm 8i$ are roots, for the particular part of the solution, we need to guess

$$\begin{aligned} y_p &= t\,(A\sin 8t + B\cos 8t) \\ y_p'' &= 2(8A\cos 8t - 8B\sin 8t) + t(-64A\sin 8t - 64B\cos 8t) \end{aligned}$$

and plugging this in gives

$$\begin{aligned} y_p'' + 64y_p &= 2(8A\cos 8t - 8B\sin 8t) + t(-64A\sin 8t - 64B\cos 8t) + 64t(A\sin 8t + B\cos 8t) \\ &= 16A\cos 8t - 16B\sin 8t \\ &= \frac{1}{24}\sin 8t + \frac{1}{16}\cos 8t \end{aligned}$$

Thus $16A = \dfrac{1}{16}$ and $-16B = \dfrac{1}{24}$, so that $A = \frac{1}{256}$ and $B = -\frac{3}{256}$. So the general solution is

$$y = c_1\cos 8t + c_2\sin 8t + \frac{t}{256}(\sin 8t - 3\cos 8t).$$

Use the initial value $y(0) = 0$:

$$y(0) = c_1 = 0.$$

Now take the derivative

$$y' = 8c_2\cos 8t + \frac{1}{256}(\sin 8t - 3\cos 8t) + \frac{t}{256}(8\cos 8t + 24\cos 8t)$$

and use the initial value $y'(0) = 0$:

$$y'(0) = 8c_2 - \frac{3}{256} = 0$$

so that

$$c_2 = -\frac{3}{2048}.$$

Thus the equation for the displacement is

$$y = -\frac{3}{2048}\sin 8t + \frac{t}{256}(\sin 8t - 3\cos 8t).$$

□

# 9. Homework 9 (4/11 & 4/13)

## Section 6.2.

**Exercise 20.** *A mass of one kg stretches a spring* 49 *cm in equilibrium. It is attached to a dashpot that supplies a damping force of* 4 *N for each m/s of speed. Find the steady state component of its displacement if it is subjected to an external force* $F(t) = 8\sin 2t - 6\cos 2t$ *N.*

*Solution.* We have $m = 1$, $c = 4$, and $\Delta l = .49$. Then $\frac{k}{m} = \frac{g}{\Delta l} = \frac{9.8}{.49} = 20$ so the differential equation we need to solve is

$$y'' + 4y' + 20y = 8\sin 2t - 6\cos 2t.$$

The steady state component will come from the particular solution only, so we will focus on finding only that piece. Since $\pm 2i$ cannot possibly be a root of the characteristic equation here, we guess

$$\begin{aligned} y_p &= A\sin 2t + B\cos 2t \\ y_p' &= 2A\cos 2t - 2B\sin 2t \\ y_p'' &= -4A\sin 2t - 4B\cos 2t \end{aligned}$$

and plug it in

$$\begin{aligned} y_p'' + 4y_p' + 20y_p &= -4A\sin 2t - 4B\cos 2t + 4(2A\cos 2t - 2B\sin 2t) + 20(A\sin 2t + B\cos 2t) \\ &= -4A\sin 2t - 4B\cos 2t + 8A\cos 2t - 8B\sin 2t + 20A\sin 2t + 20B\cos 2t \\ &= (16A - 8B)\sin 2t + (8A + 16B)\cos 2t \\ &= 8\sin 2t - 6\cos 2t \end{aligned}$$

So

$$16A - 8B = 8 \quad \textit{and} \quad 8A + 16B = -6$$

and solving this gives

$$A = \frac{1}{4} \quad \textit{and} \quad B = -\frac{1}{2}$$

so the steady state solution is

$$y = \frac{1}{4}\sin 2t - \frac{1}{2}\cos 2t.$$

□

# 10. Homework 10 (4/18 & 4/20)

## Section 11.1.

**Exercise 2.** *Solve the eigenvalue problem*

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0.$$

*Solution.* Since we know that $\lambda > 0$ (see Theorem 11.1.1), the general solution to the differential equation is

$$y = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

Using $y(0) = 0$ gives

$$y(0) = c_1 = 0$$

and using $y(\pi) = 0$ gives

$$y(\pi) = c_2 \sin(\sqrt{\lambda}\pi) = 0$$

which means either $c_2 = 0$ or $\sin(\sqrt{\lambda}) = 0$. Since $c_2 = 0$ leads to the trivial solution, we take $\sin(\sqrt{\lambda}\pi) = 0$ and get that

$$\sqrt{\lambda}\pi = n\pi \quad \Longrightarrow \quad \lambda = n^2.$$

So the eigenvalues are $\lambda_n = n^2, n = 1, 2, 3, ...$ and we then get that the corresponding eigenfunctions are

$$y_n = \sin(nx).$$

□

**Exercise 6.** *Solve the eigenvalue problem*

$$y'' + \lambda y = 0,\ y(-\pi) = y(\pi),\ y'(-\pi) = y'(\pi).$$

*Solution.* We know that $\lambda \geq 0$ are eigenvalues. For $\lambda = 0$, we get the solution

$$y = c_1 x + c_2.$$

Applying the boundary condition $y(-\pi) = y(\pi)$ gives

$$y(-\pi) = -c_1\pi + c_2 = y(\pi) = c_1\pi + c_2$$

so that we have

$$-c_1\pi = c_1\pi$$

which implies that $c_1 = 0$. Since $y' = 0$, it automatically satisfies the second boundary condition. Thus, for the eigenvalue $\lambda_0 = 0$, we have the eigenfunction $y_0 = 1$.

For $\lambda > 0$ we have the general solution

$$y = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

Applying the boundary condition $y(-\pi) = y(\pi)$ gives

$$\begin{aligned} y(-\pi) &= c_1 \cos(\sqrt{\lambda}\pi) - c_2 \sin(\sqrt{\lambda}\pi) \\ y(\pi) &= c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi) \end{aligned}$$

Equating these two gives

$$c_1 \cos(\sqrt{\lambda}\pi) - c_2 \sin(\sqrt{\lambda}\pi) = c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi)$$

or

$$c_2 \sin(\sqrt{\lambda}\pi) = 0$$

so either $c_2 = 0$ or $\sin(\sqrt{\lambda}\pi) = 0$. If we decide that $\sin(\sqrt{\lambda}\pi) = 0$, we need $\sqrt{\lambda}\pi = n\pi$ so that $\lambda = n^2$. The derivative of $y$ is

$$y' = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}t) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}t)$$

and applying the boundary condition $y'(-\pi) = y'(\pi)$ gives

$$\begin{aligned} y'(-\pi) &= \sqrt{\lambda}c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi) \\ y'(\pi) &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi) \end{aligned}$$

Equating these two gives

$$\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}\pi) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}\pi)$$

or

$$c_1 \sin(\sqrt{\lambda}\pi) = 0$$

so that either $c_1 = 0$ or $\sin(\sqrt{\lambda}\pi) = 0$. This gives us the same condition for $\lambda$ as before. Since we don't want both $c_1$ and $c_2$ equal to 0 we have two eigenfunctions for the eigenvalue $\lambda_n = n^2$:

$$y_{1n} = \cos nx \quad \textit{and} \quad y_{2n} = \sin nx.$$

Thus the complete collection of eigenvalues and eigenfunctions are

$$\begin{aligned} \lambda_0 = 1, &\quad y_0 = 1 \\ \lambda_n = n^2, &\quad y_{1n} = \cos nx \\ &\quad y_{2n} = \sin nx, \quad n = 1, 2, 3, ... \end{aligned}$$

□

**Exercise 21.** *Verify the eigenfunctions*

$$\sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, ..., \sin\frac{(2n-1)\pi x}{2L}, ...$$

*of Problem 3 are orthogonal on* $[0, L]$.

*Solution.* Recall that two functions $f(x)$ and $g(x)$ are orthogonal on $[a, b]$ if

$$\int_a^b f(x)g(x)dx = 0.$$

We need two arbitrary entries from this list:

$$\sin\frac{(2n-1)\pi x}{2L} \quad \textit{and} \quad \sin\frac{(2m-1)\pi x}{2L}.$$

We will need to use the trig identity

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$\begin{aligned}
\int_0^L \sin\frac{(2n-1)\pi x}{2L}\sin\frac{(2m-1)\pi x}{2L}dx &= \frac{1}{2}\int_0^L \left(\cos\left(\frac{(2n-1)\pi x}{2L} - \frac{(2m-1)\pi x}{2L}\right)\right. \\
&\quad \left. + \cos\left(\frac{(2n-1)\pi x}{2L} + \frac{(2m-1)\pi x}{2L}\right)\right) dx \\
&= \frac{1}{2}\int_0^L \left(\cos\left(\frac{(n-m)\pi x}{L}\right) + \cos\left(\frac{(n+m+1)\pi x}{L}\right)\right) dx \\
&= \frac{1}{2}\left(-\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\right. \\
&\quad \left. -\frac{L}{(n+m+1)\pi}\sin\left(\frac{(n+m+1)\pi x}{L}\right)\right)\Bigg|_0^L \\
&= \frac{1}{2}\left(-\frac{L}{(n-m)\pi}\sin(n-m)\pi - \frac{L}{(n+m+1)\pi}\sin(n+m+1)\pi\right) \\
&= 0
\end{aligned}$$

□

# Section 11.2.

**Exercise 3.** *Find the Fourier series of* $f(x) = 2x - 3x^2$ *on* $[-\pi, \pi]$.

*Solution.* First, compute the Fourier coefficients. Observe that $2x$ is odd and $-3x^2$ is even.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-3x^2)dx = \frac{2}{\pi}\int_0^{\pi} -3x^2 dx \\
&= \frac{2}{\pi} - x^3\Big|_0^{\pi} = -2\pi^2 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-3x^2)\cos nx\ dx = \frac{2}{\pi}\int_0^{\pi} -3x^2\cos nx\ dx \\
&= -\frac{6}{\pi}\left(\frac{1}{n}x^2\sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\ dx\right) \\
&= \frac{12}{n\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\ dx\right) \\
&= \frac{12}{n\pi}\left(-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right) \\
&= -\frac{12}{n^2}(-1)^n \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-3x^2)\sin nx\ dx = \frac{2}{\pi}\int_0^{\pi} 2x\cos nx\ dx \\
&= \frac{4}{\pi}\left(-\frac{1}{n}x\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\ dx\right) \\
&= \frac{4}{\pi}\left(-\frac{1}{n}\pi\cos n\pi + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right) \\
&= -\frac{4}{n}(-1)^n
\end{aligned}$$

Putting this together gives us the Fourier series

$$F(x) = -\pi^2 - \sum_{n=1}^{\infty} \left( \frac{12(-1)^n}{n^2} \cos nx + \frac{4(-1)^n}{n} \sin nx \right).$$

□

# 11. Homework 11 (4/25 & 4/27)

## Handout Section 2.

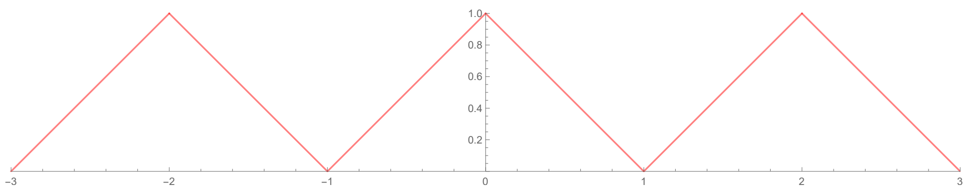
**Exercise 16.** *For the function*

$$f(x) = \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1 \end{cases}, \quad f(x+2) = f(x)$$

(a) *Sketch the graph of the given function for three periods.*
(b) *Find the Fourier series for the given function.*

*Solution.*

(a) Here is the graph over 3 periods




(b) Compute the Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{1}\int_{-1}^{1} f(x)\ dx \\ &= \int_{-1}^{0}(x+1)dx + \int_{0}^{1}(1-x)dx \\ &= 1 \\ a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\ dx \\ &= \int_{-1}^{0}(x+1)\cos n\pi x\ dx + \int_{0}^{1}(1-x) cos n\pi x\ dx \\ &= -2\frac{(-1)^n - 1}{n^2\pi^2} \\ &= \begin{cases} \dfrac{4}{n^2\pi^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \\ b_n &= \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\ dx \\ &= 0 \quad (\text{since } f(x) \text{ is even and } \sin n\pi x \text{ is odd}) \end{aligned}$$

Thus the Fourier series is

$$\begin{aligned} F(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} -2\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2}\cos(2n-1)\pi x \end{aligned}$$

$\square$

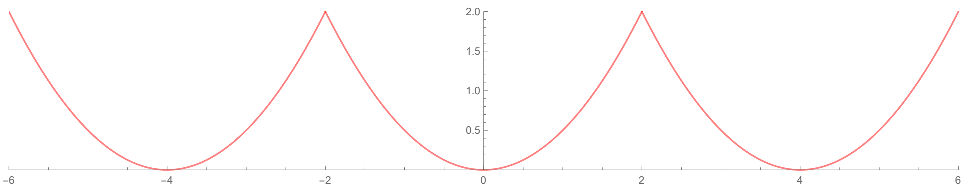
**Exercise 21.** *For the function*

$$f(x) = \frac{x^2}{2}, \ -2 \le x \le 2; \ f(x+4) = f(x)$$

(a) *Sketch the graph of the given function for three periods.*
(b) *Find the Fourier series for the given function.*
(c) *Plot $s_m(x)$ versus $x$ for $m = 5, 10, 20$.*
(d) *Describe how the Fourier series seems to be converging.*

*Solution.*

(a) Here is the graph of the function for three periods




(b) Compute the Fourier coefficients

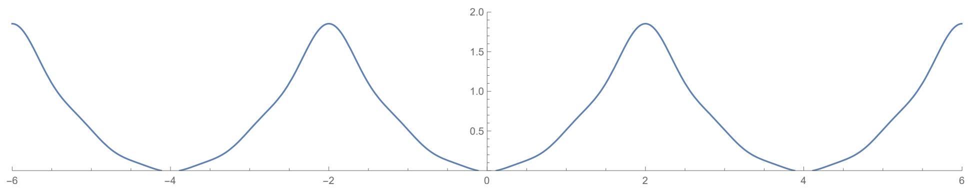
$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2} dx = \int_0^2 \frac{x^2}{2} dx \\
&= \left.\frac{x^3}{6}\right|_0^2 = \frac{8}{6} = \frac{4}{3} \\
a_n &= \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2} \cos\frac{n\pi x}{2} dx = \int_0^2 \frac{x^2}{2} \cos\frac{n\pi x}{2} dx \\
&= \frac{1}{2}\left.\left(\frac{2x^2}{n\pi}\sin\frac{n\pi x}{2} + \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right)\right|_0^2 \\
&= \frac{8}{n^2\pi^2}\cos n\pi = \frac{8(-1)^n}{n^2\pi^2} \\
b_n &= \frac{1}{2}\int_{-2}^{2} \frac{x^2}{2} \sin\frac{n\pi x}{2} dx \\
&= 0 \quad \left(\text{since } \frac{x^2}{2} \text{ is even and } \sin\frac{n\pi x}{2} \text{ is odd}\right)
\end{aligned}$$

so the Fourier series is

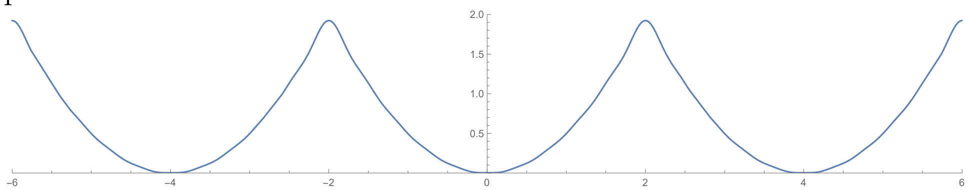
$$F(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{8(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{2}.$$

(c) The graphs $s_m(x)$ versus $x$ for:

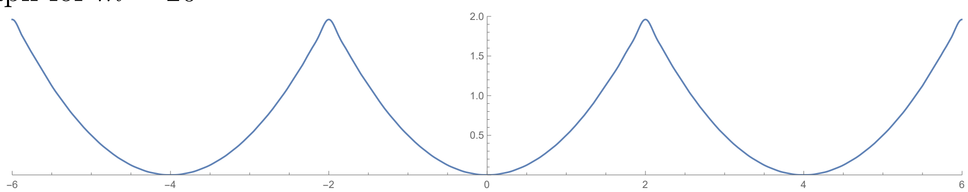
The graph for $m = 5$




The graph for $m = 10$




The graph for $m = 20$




(d) The convergence is pretty quick, except for at the endpoints of the periods where the original function has sharp corners ($x = ..., -6, -2, 2, 6, ...$)

□

# Handout Section 3.

**Exercise 17.** *Assuming that*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

*show formally that*

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right).$$

*This relation between a function $f$ and it's Fourier coefficients is known as* Parseval's equation. *This relation is very important in the theory of Fourier series.*

*Proof.* Take

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

and multiply though by $f(x)$:

$$[f(x)]^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} \left( a_n f(x) \cos \frac{n\pi x}{L} + b_n f(x) \sin \frac{n\pi x}{L} \right)$$

and integrate both sides with respect to $x$ from $-L$ to $L$:

$$\int_{-L}^{L} [f(x)]^2 \, dx = \int_{-L}^{L} \frac{a_0}{2} f(x) \, dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left( a_n f(x) \cos \frac{n\pi x}{L} + b_n f(x) \sin \frac{n\pi x}{L} \right) dx$$

Formally, we may pass the integral though the infinite sum to get

$$\begin{aligned}
\int_{-L}^{L} [f(x)]^2\, dx &= \int_{-L}^{L} \frac{a_0}{2} f(x)\ dx + \sum_{n=1}^{\infty} \left( \int_{-L}^{L} a_n f(x) \cos\frac{n\pi x}{L}\ dx + \int_{-L}^{L} b_n f(x) \sin\frac{n\pi x}{L}\ dx \right) \\
&= \frac{a_0}{2} \int_{-L}^{L} f(x)\ dx + \sum_{n=1}^{\infty} \left( a_n \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\ dx + b_n \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\ dx \right) \\
&= \frac{a_0}{2}(La_0) + \sum_{n=1}^{\infty} \left( a_n(La_n) + b_n(Lb_n) \right) \\
&= L\left( \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right) \right)
\end{aligned}$$

Divide both sides by $L$ to get the desired equation

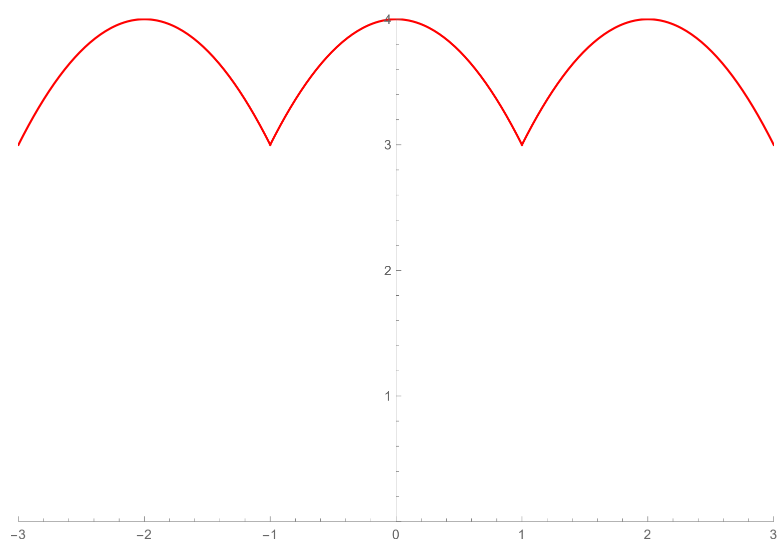
$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right).$$

□

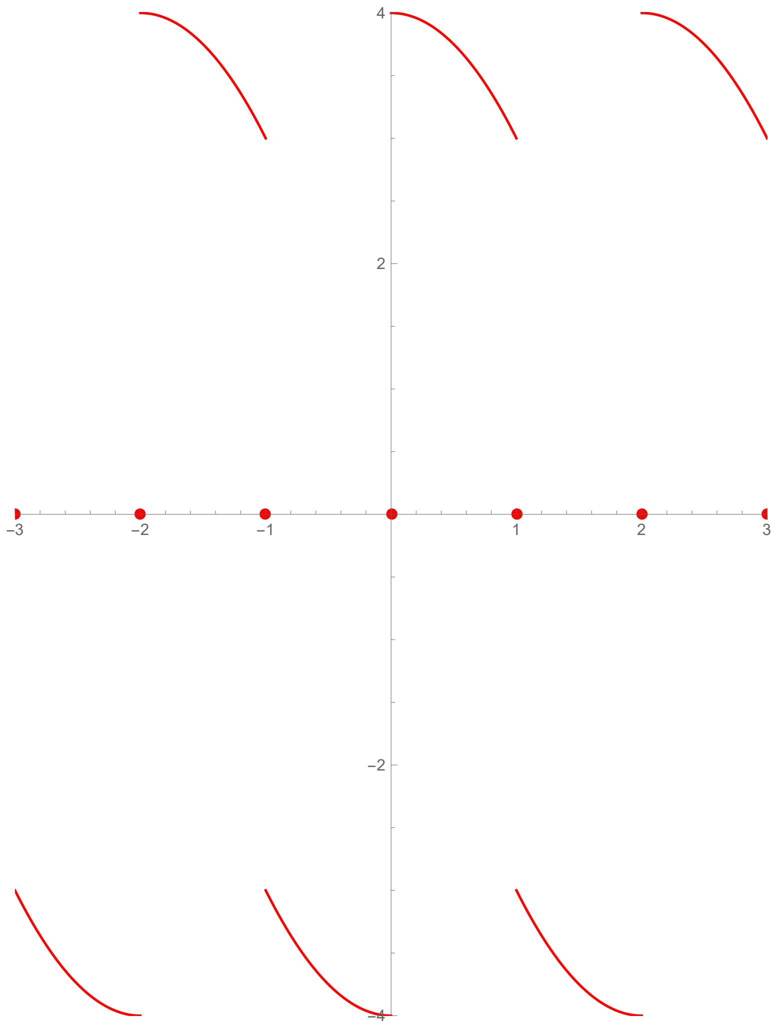
# Handout Section 4.

**Exercise 12.** *Sketch the graphs of the even and odd extensions of $f$ of period 2 where $f(x) = 4 - x^2$, $0 < x < 1$.*

*Solution.* The even extension is



The odd extension is




□

**Exercise 13.** *Prove that any function can be expressed as the sum of two other functions, one of which is even and the other odd. That is, for any function $f$, whose domain contains $-x$ whenever it contains $x$, show that there are an even function $g$ and an odd function $h$ such that $f(x) = g(x) + h(x)$.*

*Proof.* Define the function $a(x) = f(x) + f(-x)$ and note that $a(-x) = f(-x) + f(x) = a(x)$ so that $a(x)$ is an even function. Define also the function $b(x) = f(x) - f(-x)$ and note that $b(-x) = f(-x) - f(x) = -b(x)$ so that $b(x)$ is an odd function. Observe that $a(x) + b(x) = f(x) + f(-x) + f(x) - f(-x) = 2f(x)$ and define

$$g(x) = \frac{f(x) + f(-x)}{2} \quad \textit{and} \quad h(x) = \frac{f(x) - f(-x)}{2}.$$

Then $g(x)$ is even, $h(x)$ is odd, and $f(x) = g(x) + h(x)$. □

**Exercise 23.** *For the function*

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

(a) *Find the cosine series of period $4\pi$ for $f(x)$.*
(b) *Sketch the graph of the function to which the series converges for three periods.*
(c) *Plot one or more partial sums of the series.*

*Solution.*

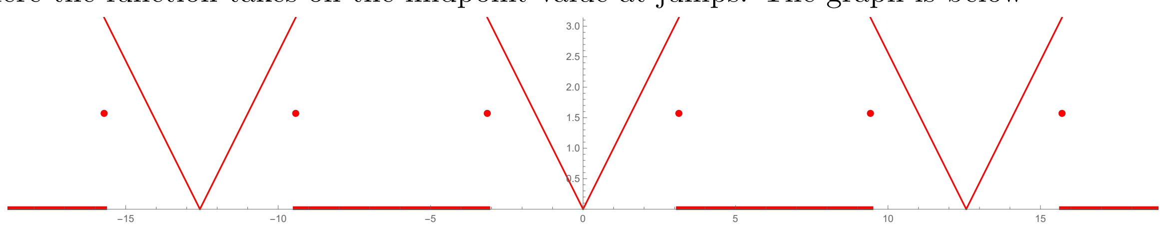
(a) When finding the cosine series, the coefficients are

$$\begin{aligned} a_0 &= \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx \\ &= \frac{1}{\pi}\frac{1}{2}x^2\Big|_0^{\pi} = \frac{\pi}{2} \\ a_n &= \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\frac{n\pi x}{2\pi}\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos\frac{nx}{2}\,dx \\ &= \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + n\pi\sin\frac{n\pi}{2} - 2\right) \end{aligned}$$

so the Fourier cosine series is

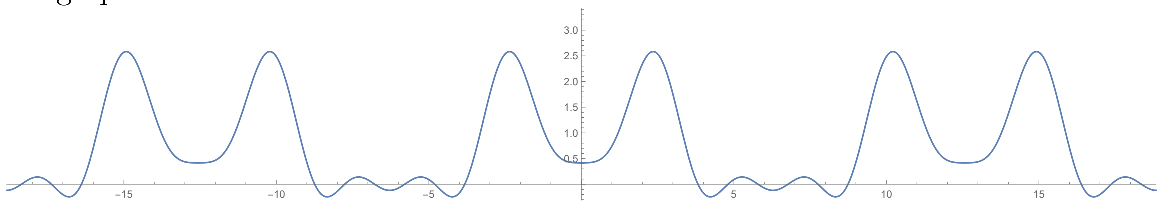
$$F(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + n\pi\sin\frac{n\pi}{2} - 2\right)\cos\frac{nx}{2}.$$

(b) Since we found the cosine series for $f(x)$, the series converges to the even extension of $f(x)$, where the function takes on the midpoint value at jumps. The graph is below

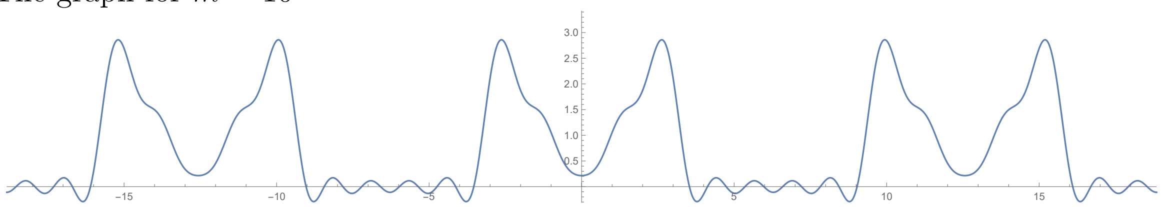


(c) Below are graphs of partial sums, $s_m(x)$ of the Fourier series, $F(x)$:

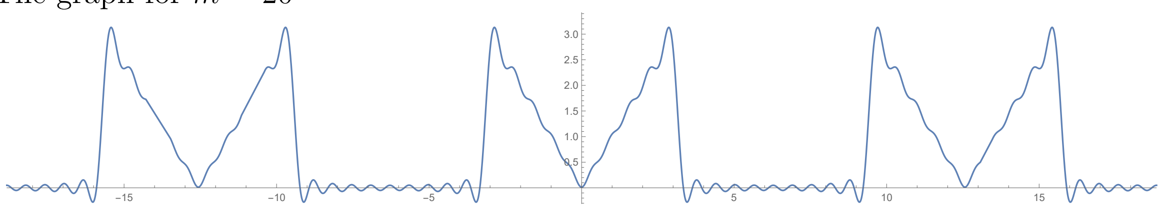
The graph for $m = 5$



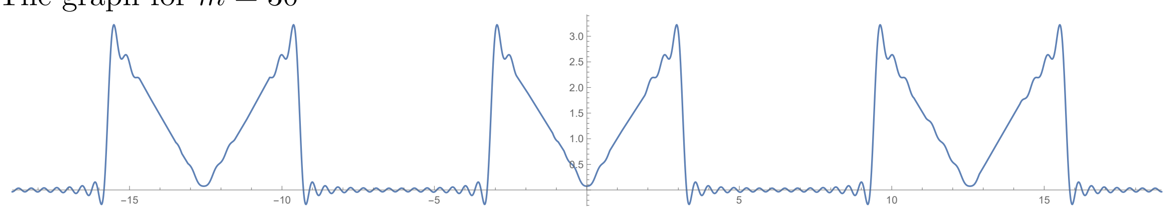

The graph for $m = 10$



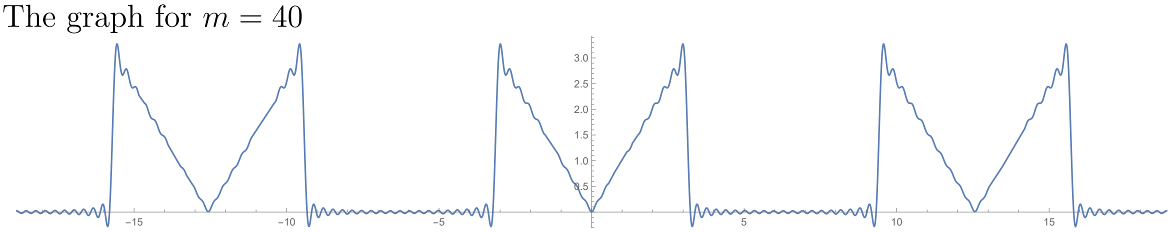

The graph for $m = 20$



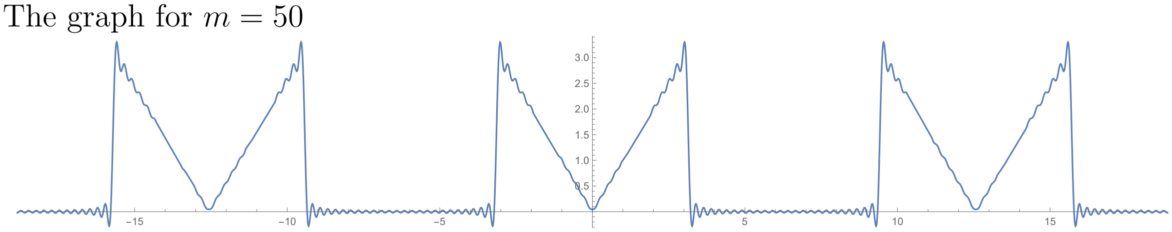

The graph for $m = 30$




The graph for $m = 40$




The graph for $m = 50$




The Gibbs Phenomenon can be seen pretty clearly in the larger partial sums and how the spike happens closer and closer to the jump in the function.

□

# 12. Homework 12 (5/2 & 5/4)

## Section 12.1.

**Exercise 8.** *Solve the heat equation with the given boundary and initial conditions*

$$\begin{array}{l} u_t = u_{xx},\ \ 0 < x < 1,\ \ t > 0, \\ u(0,t) = 0,\ \ u(1,t) = 0,\ \ t > 0, \\ u(x,0) = x(1-x),\ \ 0 \le x \le 1 \end{array}$$

*Solution.* In this problem, $a^2 = 1$ and $L = 1$. The solution of the heat equation with these boundary values takes on the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where $c_n$ are the Fourier sine series coefficients of $u(x,0)$. That is

$$\begin{aligned} c_n &= \frac{2}{1}\int_0^1 x(1-x)\sin n\pi x\ dx = 2\int_0^1 (x-x^2)\sin n\pi x\ dx \\ &= 2\left( \left.\frac{(-x+x^2)\cos n\pi x}{n\pi}\right|_0^1 + \frac{1}{n\pi}\int_0^1 (2x-1)\cos n\pi x\ dx \right) \\ &= \frac{2}{n\pi}\left( \left.\frac{(2x-1)\sin n\pi x}{n\pi}\right|_0^1 - \frac{2}{n\pi}\int_0^1 \sin n\pi x\ dx \right) \\ &= \left.\frac{4}{n^3\pi^3}\cos n\pi x\right|_0^1 \\ &= \frac{4}{n^3\pi^3}(\cos n\pi - 1) = \frac{4}{n^3\pi^3}((-1)^n - 1) \\ &= \begin{cases} -\dfrac{8}{n^3\pi^3}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{8}{(2n-1)^3\pi^3} e^{-(2n-1)^2\pi^2 t} \sin{(2n-1)\pi x}.$$

□